Asymptotic states in long Josephson junctions in an external magnetic field

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Abstract

Asymptotic states in long Josephson junctions are investigated in an external magnetic field. We show that a choice one of the solution of the stationary Ferrell-Prange equation can carry be out with use of an asymptotic solution of the sine-Gordon equation and that an evolution to that stable solution occurs by passing through metastable states, which is determined with a form of quickly damped initial perturbation. The boundary sine-Gordon and Ferrell-Prange problems were carried out with a numerical simulation. An approximated expression for the vortex and antivortex states is obtained in the case of large values of an external magnetic field.

The problem of magnetic field penetration in long Josephson junctions is in a sense a classical one [1]–[5]. Nevertheless, interest in it has not abated [6]–[13]. For example, in long Josephson junctions a dynamical chaos was discovered [6]–[10]. A long Josephson junction is a very good system to study Josephson vortex (fluxons, solitons) motions and vortex interactions [12]–[13].

It is well known that a weak magnetic field penetrates exponentially in a long Josephson junction. In a strong magnetic field a state of the long Josephson junction arises that is characterized by the appearance of a Josephson vortex. An equation that describes this phenomenon is nonlinear and in a general case it is the driven damped sine-Gordon equation. In a stationary case we have the nonlinear Ferrell-Prange equation and one would think

the stationary states of a long Josephson junction represent solutions of the above equation, although it turned out that the Ferrell-Prange equation has more than one solution and the number of them increases by increasing the total length L of the long Josephson junction. Consequently, the problem of the selection of solutions arises. Considering this problem from the thermodynamic point of view one can affirm that a solution is realized which corresponds to a minimum of the thermodynamic potential. However, the situation turns out to be more complicated. In fact, if we suppose that the minimum of the thermodynamic potential is satisfied with some solutions of the stationary equation, then this criterion for selection of the solutions turns out to be insufficient. Apparently this is precisely what takes place in our case. In the present work it is shown that the problem of selection of solutions for the stationary Ferrell-Prange equation is unequivocally solved in the following way: An asymptotic solution of the nonstationary sine-Gordon equation is found that coincides with the solution of the stationary Ferrell-Prange equation by $t \to \infty$, and what is more either of asimptotic solution, which is realized, depends on a form of rapid damped initial perturbation or on a way of switching on an external magnetic field. This exceptionally important role of initial perturbation that appears as a trigger mechanism is obviously explained by system nonlinearity.

We consider a long Josephson junction in an external magnetic field H_{ext} perpendicular to the junction. The Josephson phase variable $\varphi(x,t)$ in a long Josephson junction is satisfied with the one-dimensional sine-Gordon equation

$$\varphi_{tt}(x,t) + 2\gamma\varphi_t(x,t) - \varphi_{xx}(x,t) = -\sin\varphi(x,t), \tag{1}$$

where x is the distance along the junction normalized to the Josephson penetration length λ_J,

$$\lambda_J = \Big(\frac{c\Phi_0}{8\pi^2 j_c d}\Big)^{1/2},$$

t is the time normalized to the inverse of the Josephson plasma frequency ω_J,

$$\omega_J = \Big(\frac{2\pi c j_c}{C\Phi_0}\Big)^{1/2},$$

γ is the dissipative coefficient per unit area, Φ_0 is the flux quantum, j_c is the ctitical current density of superconductor, $d = 2\lambda_L + b$, λ_L is the London

penetration length,b is the thickness of the dielectric barrier, C is the junction capacitance per unit area.

Conditions corresponding to the presence of the external magnetic field can be modeled by the following boundary conditions:

$$\varphi_x(x,t)|_{x=0} = \varphi_x(x,t)|_{x=L} = H_{ext}(0,t) = H_{ext}(L,t). \tag{2}$$

Here and below the magnetic field is normalized to $\Phi_0/2\pi\lambda_J d$, and the total length of the junction L is normalized to λ_J.

First of all we consider an asimptotic solution (by $t \to \infty$) of Eq. (1) with the boundary conditions (2) and suppose that $\lim_{t\to\infty} H_{ext}(0,t) = H_0$. At first sight this problem is trivial, since all asymptotic solutions will be stationary on account of dissipation, they will be satisfied with the stationary Ferrell-Prange equation with the corresponding boundary conditions:

$$\varphi_{0xx}(x) = \sin\varphi_0(x), \varphi_{0x}(x)|_{x=0} = \varphi_{0x}(x)|_{x=L} = H_0. \tag{3}$$

However, numerical simulations of the boundary problem (1),(2) and (3) show that not all solutions of the problem (3) are asymptotic solutions of the problem (1), (2) or in other words we can say that not all solutions of the problem (3) are physically observed. For example, for the small external fields $H_0 < 1$ there exists a solution of the problem (3) that corresponds to an observed dumping of the magnetic field into a junction. Simultaneously with that solution another solution of the problem (3) exists that gives very strong growth of a field in the same direction into a long Josephson junction. To cut off "nonphysical" solutions supplementry considerations are attracted in the form of a condition of the minimum of a thermodynamic potential, as was mentioned above.

The results of problem (1), (2) simulations showed that "nonphysical" solutions are never realized in the asymptotic approximation. The reason for the lack of the asymptotic "nonphysical" solutions is their instability relative to small perturbations.

We show that the stationary solution $\varphi_0(x)$ of the problem (3) is stable then and only then when all values of the spectrum E are positive, i.e., $E_{\min} > 0$, for the boundary problem

$$\begin{gathered} -u_{xx}(x) + u(x)\cos\varphi_0(x) = Eu(x), \\ u_x(x)|_{x=0} = u_x(x)|_{x=L} = 0. \end{gathered} \tag{4}$$

To prove this assertion we shall linearize the sine-Gordon equation in the vicinity of stationary solution $\varphi_0(x)$; i.e., we shall suppose $\varphi(x,t) = \varphi_0(x) + \theta(x,t)$, where $\theta(x,t)$ is an infinitesimal perturbation. We obtain the equation for the function $\theta(x,t)$ from the sine-Gordon equation taking into account the Ferrell-Prange equation:

$$\begin{gathered}\theta_{tt}(x,t) + 2\gamma\theta_t(x,t) - \theta_{xx}(x,t) = -\theta(x,t)\cos\varphi_0(x), \\ \theta_x(x)|_{x=0} = \theta_x(x)|_{x=L} = 0.\end{gathered} \tag{5}$$

We can obtain a general solution of the boundary Eq. (5) by means of the expansion of the function $\theta(x,t)$ in a series in terms of a complete system of eigenfunctions of the Schrödinger operator with the potential $\cos\varphi_0(x)$:

$$\theta(x,t) = \sum_n e^{\lambda_n t} u_n(x), \tag{6}$$

where λ_n and $u_n(x)$ are eigenvalues and eigenfunctions of the Schrödinger operator of the problem (4), respectively. Substituting expansion (6) into Eq. (5) and taking into account Eq. (4), we get for λ_n:

$$\lambda_n = -\gamma \pm \sqrt{\gamma^2 - E_n}. \tag{7}$$

It is clear that if even one of the numbers λ_n has a positive real part, then the infinitesimal perturbations $\theta(x,t)$ will increase exponentially in the course of time and, further, if all λ_n have a negative real part, then the perturbations damp exponentially. Since the Schrödinger operator is a Hermitian operator in a space of functions which have a derivative on the ends of intervals equal to zero, its eigenvalues E_n are real. It results from expression (7) that λ_n can have a positive real part only with $E_n < 0$. Thus with $E_n > 0$ all $\lambda_n < 0$ and perturbations $\theta(x,t)$ damp exponentially. It is clear that the condition $E_{n\min} > 0$ is the condition of the stable solution of the problem (3) with any values $\gamma > 0$.

One can call unstable solutions $\varphi_0(x)$ of the problem (3) metastable, since they may be long-lived. Analysis of the spectrum problem gives information about the survival time of metastable states. Let $\Re\lambda_n > 0$; then $\tau = (\Re\lambda_n)^{-1}$ makes sense of the characteristic decay time of the metastable state. The corresponding eigenfunctions $u_n(x)$ are perturbations that destroy the metastable states.

A detailed analysis of the problem (1), (2), (3), and (4) is failed to be carried out by analytical methods. The numerical simulation results of this problems showed the following.

(1) The number N of stable and metastable states, i.e., the number of solutions of the problem (3), increases with increasing L. For example, with $H_0 > 2$ for $L = 2\pi$, $N = 4$, for $L = 20$, $N = 8$.

(2) From all solutions of the problem (3) approximately half of the states is stable. For example, with $H_0 = 3$ for $L = 2\pi$, 2 states out of 4 are stable; for $L = 20$, 3 states out of 8 are stable; for $L = 40$, 11 states of 18 are stable.

(3) The numerical investigations of the sine-Gordon Eq. (1) showed that either stable state [stable solution of the problem (3)] is realized depending on the method of superposition of an external field in a long Josephson junction. In other words, an established magnetic field and currents in the junction "remember" how an external magnetic field was started.

For the case $H_0 = 2$, $L = 2\pi$ switching on the magnetic field in the form $H_{\text{ext}}(0, t) = H_{\text{ext}}(L, t) = H_0[1 - a \exp(-t/5) \cos t]$ leads to the system coming to one or another state (Fig. 1) depending on the controlling parameter a. So with $0 \le a \le 0.36$ state 4 is realized in Fig. 1; with $0.37 \le a \le 0.74$ the one fluxon state 6 is realized; with $a \ge 0.75$ the two fluxon state 8 is realized.

It turns out that transition to one of the stable asymptotic states occurs through the metastable states. For example, with $a = 0.74$ state 6 arises from the metastable state 7; with $a = 0.75$ state 8 arises from the same metastable state 7 (Fig. 1). This property of the system is not characteristic for the case $H_0 = 2$, corresponding to a separatrix. For other values of H_0 the same property takes place. In Fig. 2 the evolution of the magnetic field is shown for the case $H_0 = 2$ with different values of the parameter a into the junction.

In our opinion the conclusion is the most important. Thus, an asymptotic state of a magnetic field, realized in a long Josephson junction depends essentially on a method of switching on an external field, which is a direct result of the sine-Gordon equation. The asymptote of problem (1), (2) taking into account the method of switching on an external field does select one of the stable solutions of the stationary Ferrell-Prange equation. It is possible that this property of the solution is general enough for nonlinear systems. With the condition of minimum thermodynamic potential all stable solutions of problem (3) are satisfied apparently.

We should like to note another interesting fact following from the nu-

merical simulation of problems (1), (2) and (3). Among all the stationary solutions there exists one which corresponds to the maximum of a magnetic field and the another that corresponds to the minimum of a magnetic field. In Fig. 1 these are states 8 and 6, respectively. The states are stable. State 8 in this figure is called a vortex; state 6 in the same figure is called an antivortex. For these states we can obtain an approximate formula solving problem (3) by means of the iteraction method

$$\begin{aligned} &\varphi_0(x) \equiv \lim_{k\to\infty} \varphi_0^{(k)}(x), \\ &\varphi_{0xx}^{(k)}(x) = \sin \varphi_0^{(k-1)}(x), \\ &\varphi_{0x}^{(k)}|_{x=0} = \varphi_{0x}^{(k)}|_{x=L} = H_0. \end{aligned} \tag{8}$$

Suppose $\varphi_0^{(0)}(x) = H_0 x + \alpha$, where α is some kind of constant. Then

$$\begin{aligned} &\varphi_{0xx}^{(1)}(x) = \sin(H_0 x + \alpha), \\ &\varphi_{0x}^{(1)}|_{x=0} = \varphi_{0x}^{(1)}|_{x=L} = H_0. \end{aligned} \tag{9}$$

Integrating (8), we find

$$H^{(1)} \equiv \varphi_{0x}^{(1)}(x) = H_0 \pm \frac{1}{H_0}\Big[\cos(H_0 x - \frac{H_0 L}{2}) - \cos(\frac{H_0 L}{2})\Big]. \tag{10}$$

In formula (10) the "+" sign corresponds to a vortex state; the "−" sign corresponds to an antivortex state. Expression (10) is correct in the case of large fields. Comparision of the values, obtained by means of formula (10), with results of numerical simulation shows that they practically do not differ for $H_0 = 10$.

In conclusion we note that a swing of deflections of the field $H^{(1)}(x)$ from H_0 makes $\approx 2/H_0$ and this swing decrease by increasing of the external field. In other words one can say that the number of the stationary states per unit of field increases. In turn this circumstance may lead to the small fluctuations of the external field H_0 to cause of a "jump" from the one stable state to the other. We hope to discuss this problem in detail in our next work.

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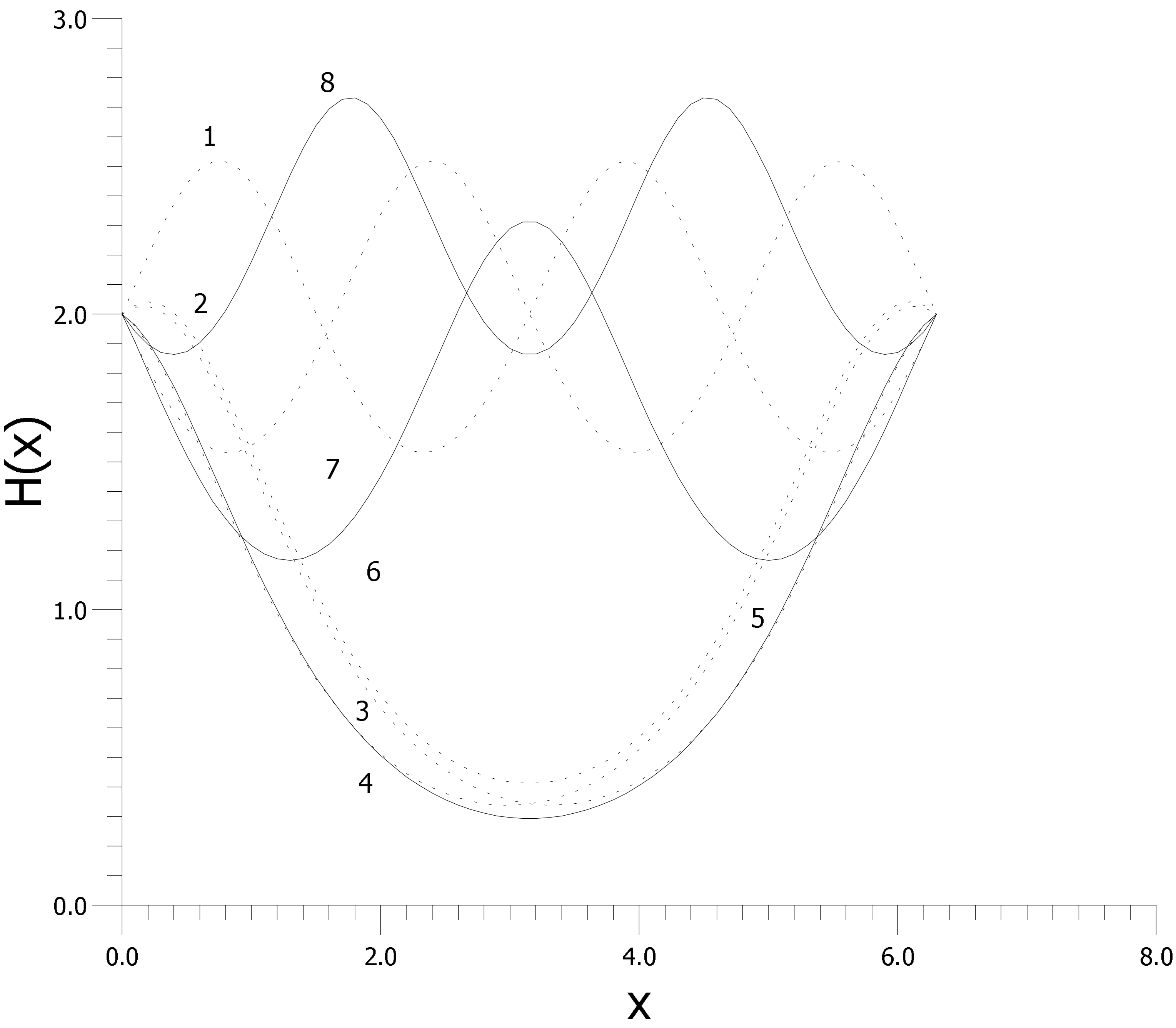
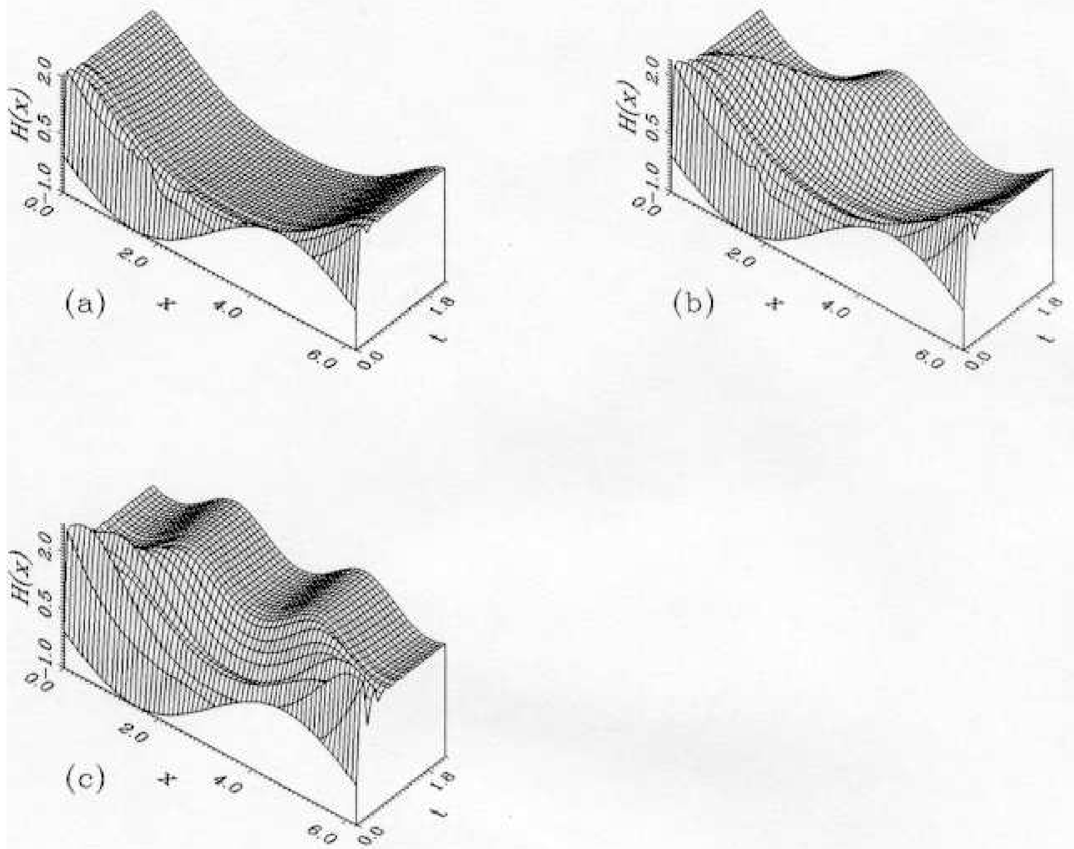
Fig. 1. Solutions of boundary problem (3) for the case $H_0 = 2, L = 2\pi$. Solid line (4,6,8), stable states; dashed line (1,2,3,5,7), metastable states.

Fig. 2. Evolution of magnetic field for the case $H_0 = 2, L = 2\pi$ with different values of parameter a. (a) $a = 0.2$, (b) $a = 0.5$, and (c) $a = 1.0$.



H(x)
x
3.0
2.0
1.0
0.0
0.0
2.0
4.0
6.0
8.0
1
2
3
4
5
6
7
8


(a)
(b)
(c)
H(x)
x
t
0.0
0.5
1.0
2.0
4.0
6.0
1.8